

# Off-shell tachyon amplitudes: analyticity and projective invariance<sup>1</sup>

Francesco Cuomo<sup>a</sup>, Raffaele Marotta<sup>b</sup>, Francesco Nicodemi<sup>a</sup>,  
Roberto Pettorino<sup>a</sup>, Franco Pezzella<sup>a</sup> and Gianluca Sabella<sup>a</sup>

<sup>a</sup> *Dipartimento di Scienze Fisiche, Università di Napoli "Federico II"*  
*and I.N.F.N., sezione di Napoli*

*Complesso Universitario di Monte S. Angelo, Via Cintia, I-80126 Napoli, Italy*

*name.surname@na.infn.it*

<sup>b</sup> *NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

*name.surname@nordita.dk*

## Abstract

We compute off-shell three- and four-tachyon amplitudes at tree level by using a prescription based on the requirement of projective invariance. In particular we show that the off-shell four tachyon amplitude can be put in the same form as the corresponding on-shell one, exhibiting therefore the same analyticity properties. This is shown both for the bosonic and for the fermionic string. The result obtained in the latter case can be extended to the off-shell four-tachyon amplitude in type 0 theory.

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# 1 Introduction

The presence of the tachyon in the spectrum of string theories has been always considered as a disease even though it was generally seen as a sign of a wrong choice of the vacuum. Superstring theory, while on the one hand has solved the problems of the physical spectrum projecting out tachyons, on the other hand has obscured the interest in the dynamics of these particles. In the last years the proposal by A. Sen [1] of a well defined mechanism for tachyon condensation has attracted wide interest [2]. The main ingredient of this scenario is the tachyon potential which depends on the formulation of the off-shell theory. Several different approaches have been followed in order to provide such an extension (see e.g. [3]), many of which in the context of string field theory [4]. We think that some help in understanding these features may be provided by the operator formalism of the  $N$ -string Vertex [5] for evaluating tachyon amplitudes suitably extended off-shell.

In some recent papers [6] [7] prescriptions have been given in order to compute off-shell string scattering amplitudes in the above mentioned formalism. In particular in [7] it has been proposed a prescription based on the property of projective invariance that must be exhibited both by on- and off-shell amplitudes. Their being projective invariant is crucial if factorization is required to hold [8]. Such a requirement inspires the right choice for the local coordinate systems defined around the punctures of the external states. In fact the  $N$ -string Vertex is an operator which depends on  $N$  complex Koba-Nielsen variables, corresponding to those punctures, through  $N$  projective transformations  $V_i(z)$  which define local coordinate systems vanishing around each  $z_i$ , i.e.

$$V_i(0) = z_i.$$

The  $N$ -string Vertex can be regarded as a sort of functional generator of scattering amplitudes among arbitrary string states. When it is saturated with  $N$  physical on-shell states, the corresponding amplitude is independent of the  $V_i$ 's while, when saturated on off-shell states, its dependence on these maps is transferred to the amplitudes themselves. Hence, on-shell string amplitudes are independent of the choice of such local coordinate

systems around the punctures and are described in terms of correlation functions of vertex operators which are primary fields of the underlying conformal field theory. On the contrary, off-shell string amplitudes depend on the choice of the local coordinate systems and are given by correlation functions of vertex operators corresponding to quasi-primary fields. There is an analogy with gauge theories. Indeed choosing  $V_i(z)$  is equivalent to perform a gauge choice, since on-shell amplitudes are gauge invariant, while their off-shell counterparts are not.

In the case of off-shell string amplitudes not all the choices of the  $V_i$ 's are equivalent; requiring their projective invariance suggests the suitable ones.

In this paper, following the above prescription, we compute off-shell amplitudes involving three and four tachyons starting from the bosonic string case. In particular we show that in the case of open strings, the off-shell amplitude can be put in the same form as its on-shell counterpart, so recovering the same analytic properties. This result will be shown to be valid also for the closed string if one enforces the kinematic condition

$$s + t + u = 4m^2 \tag{1.1}$$

where  $s = -(p_1 + p_2)^2, t = -(p_1 + p_4)^2, u = -(p_1 + p_3)^2$  are the usual Mandelstamm variables and  $m$  is the tachyon mass, but  $p_i^2$  is unconstrained.

The computation has been then extended to the fermionic string case and of course limited to the four-tachyon amplitude, since the three-tachyon one is vanishing. In this case one has to take into consideration a dependence of the off-shell amplitudes on the *pictures* which must be assigned to the tachyons in order to saturate the ghost number conservation on the sphere. Projective invariance again inspires the choice of the local coordinate systems around the punctures. We show that while for a given picture assignment the off-shell amplitudes depend on the performed choice, the average over all possible picture assignments yields an off-shell four-tachyon amplitude of the same form as the on-shell one. Also this result follows after imposing the kinematic constraint (1.1).

The paper is organized as follows.

In sect 2. we report the results relative to the off-shell three and four-tachyon ampli-

tudes in the bosonic open and closed string theory.

In sect. 3 we apply the same procedure to the four-tachyon amplitude in the fermionic string case. We notice that such a computation is the same as in type 0 theory.

## 2 Three- and four-tachyon bosonic string amplitudes

### 2.1 Open bosonic string

We start from the  $N$ -string 0-loop vertex  $\widehat{V}_{N;0}^{op}$  generating the tree diagrams of oriented open strings [10] and specialized to the case of  $N$  tachyons. It assumes the following form:

$$\widehat{V}_{N;0}^{op} = C_0^{op} \langle \Omega | \int [dm]_N^0 \prod_{i=1}^N \exp \{ \alpha' \ln V'_i(0) \hat{p}_i^2 \} \prod_{\substack{i,j=1 \\ i < j}}^N \exp \{ 2\alpha' \lg(z_i - z_j) \hat{p}_i \cdot \hat{p}_j \}, \quad (2.1)$$

where  $C_0^{op}$  is a normalization factor given, in  $d$  dimensions, by [11]:

$$C_0^{op} = g_o^{-2} \frac{1}{(2\alpha')^{d/2}}, \quad (2.2)$$

$g_o$  being the open string coupling constant. The bra  $\langle \Omega | \equiv \Pi_{i=1}^N \langle x_i = 0 | \delta \left( \sum_{i=1}^N \hat{p}_i \right)$  represents the product of the vacua of the Fock spaces of each tachyon. The measure is defined by

$$[dm]_N^0 = \frac{1}{dV_{abc}} \prod_{i=1}^{N-1} [\vartheta(z_i - z_{i+1})] \prod_{i=1}^N \frac{dz_i}{V'_i(0)} \quad (2.3)$$

and  $dV_{abc}$  is the projective invariant volume element

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_a - z_b)(z_b - z_c)(z_a - z_c)}.$$

Let us denote by  $|p\rangle$  a tachyon state with momentum  $p$ . It is created by the vertex operator

$$\mathcal{V}(z) = \mathcal{N}_t : e^{i\sqrt{2\alpha'} p \cdot X(z)} :$$

where the colons denote the standard normal ordering on the modes of the open string coordinate  $X^\mu(z)$  and  $\mathcal{N}_t$  is a normalization factor [11]:

$$\mathcal{N}_t = 2g_o (2\alpha')^{(d-2)/4}.$$

If we write, as usual,

$$X^\mu(z) = \hat{x}^\mu - i\hat{p}^\mu \log z + i \sum_{n \neq 0} \frac{\hat{\alpha}^\mu}{n} z^{-n}$$

then the tachyon state is

$$|p\rangle \equiv \lim_{z \rightarrow 0} \mathcal{V}_p(z) |0; p=0\rangle = \mathcal{N}_t e^{ip \cdot \hat{x}} |0; p=0\rangle.$$

The tachyon is on-shell if

$$p^2 = -m^2 = \frac{1}{\alpha'}.$$

Saturating  $\hat{V}_{N;0}^{op}$  with  $N$  off-shell tachyons, with momenta  $p^2 \neq \frac{1}{\alpha'}$ , one gets the following contribution to the off-shell  $N$ -tachyon amplitude:

$$\begin{aligned} \mathcal{A}_N^{op}(p_1, \dots, p_N) &= C_0^{op} \mathcal{N}_t^N \int \frac{1}{dV_{abc}} \prod_{i=1}^N \left[ dz_i (V_i'(0))^{\alpha' p_i^2 - 1} \right] \\ &\times \prod_{i=1}^{N-1} \vartheta(z_i - z_{i+1}) \prod_{\substack{i,j=1 \\ i < j}}^N (z_i - z_j)^{2\alpha' p_i \cdot p_j}. \end{aligned} \quad (2.4)$$

The total amplitude is to be obtained by summing over non-cyclic permutations of the external states.

### 2.1.1 Projective invariance

While for on-shell amplitudes the dependence on the local coordinate systems  $V_i$  clearly disappears, this isn't true for the off-shell ones, which depend on them. Requiring projective invariance for the off-shell amplitudes inspires the choice of the  $V_i$ 's. Under an  $SL(2, \mathcal{R})$  transformation

$$z \rightarrow z' = \frac{az + b}{cz + d} \quad ad - bc \neq 0 \quad a, b, c, d \in \mathcal{R},$$

the quantities  $V'_i(0)$  have to transform according to

$$V'_i(0) \rightarrow V'_i(0) \frac{ad - bc}{(cz_i + d)^2}.$$

A suitable choice is then

$$V'_i(0) = \frac{(z_{i-1} - z_i)(z_i - z_{i+1})}{(z_{i+1} - z_{i-1})} \rho(z_1, \dots, z_N), \quad (2.5)$$

with  $z_0 = z_N$ ,  $z_{N+1} = z_1$  and  $\rho$  any projective invariant function of the punctures.

In the following we will choose  $\rho = 1$ , in which case the expression in (2.5) coincides with the first derivative evaluated in  $z = 0$  of the so-called Lovelace function used in the dual models [12]. Under an  $SL(2, \mathcal{R})$  transformation, which preserves the cycling ordering, the other factors in the amplitude (2.4) transform as

$$\begin{cases} dz_i \rightarrow \frac{ad-bc}{(cz_i+d)^2} dz_i \\ z_i - z_j \rightarrow \frac{ad-bc}{(cz_i+d)(cz_j+d)} (z_i - z_j) \\ dV_{abc} \rightarrow dV_{abc} \end{cases} \quad (2.6)$$

Therefore the integrand  $\mathcal{I}$  in the amplitude transforms according to

$$\mathcal{I} \rightarrow \prod_{i=1}^N \left\{ \left[ \frac{ad-bc}{(cz_i+d)^2} \right]^{\alpha' p_i^2 - 1} \times \left[ \frac{ad-bc}{(cz_i+d)^2} \right]^{-\alpha' p_i^2} \right\} \mathcal{I},$$

while the measure transforms as

$$\prod_{i=1}^N dz_i \rightarrow \prod_{i=1}^N \left[ \frac{ad-bc}{(cz_i+d)^2} \right] dz_i,$$

showing that the off-shell amplitude is projective invariant.

In computing amplitudes we are therefore allowed to fix three of the punctures at three specific points on the real axis.

### 2.1.2 Three- and four-tachyon amplitudes

For the three-tachyon case the integral in (2.4) is just one and the amplitude turns out to be the product of  $C_0^{op}$  by  $\mathcal{N}_t^3$ , independently of the tachyon momenta<sup>2</sup>, i.e.:

$$\mathcal{A}_3 = 8g_o(2\alpha')^{(d-6)/4}. \quad (2.7)$$

In the case of the four-tachyon amplitude, if we fix the Möbius gauge as  $z_{1,2,4} = \infty, 1, 0$  the contribution to the amplitude results to be

$$\begin{aligned} \mathcal{A}_4^{op} &= C_0^{op} \mathcal{N}_t^4 \int_0^1 dz z^{-\alpha' s - 2} (1 - z)^{-\alpha' t - 2} \\ &= 16g_o^2 (2\alpha')^{(d-4)/2} B(-\alpha' s - 1, -\alpha' t - 1). \end{aligned} \quad (2.8)$$

The total off-shell four-tachyon amplitude is obtained by considering the sum over the independent non cyclic permutations:

$$\begin{aligned} \mathcal{A}_4^{op}(s, t, u) &= 16g_o^2 (2\alpha')^{(d-4)/2} \left\{ \frac{\Gamma[-\alpha(s)]\Gamma[-\alpha(t)]}{\Gamma[-\alpha(s) - \alpha(t)]} + \frac{\Gamma[-\alpha(s)]\Gamma[-\alpha(u)]}{\Gamma[-\alpha(s) - \alpha(u)]} \right. \\ &\quad \left. + \frac{\Gamma[-\alpha(t)]\Gamma[-\alpha(u)]}{\Gamma[-\alpha(t) - \alpha(u)]} \right\}, \end{aligned} \quad (2.9)$$

being  $\alpha(x) = \alpha'x + 1$  the usual Regge trajectory. This expression is identical to the Veneziano amplitude, but here the tachyon momenta are only restricted to verify momentum conservation.

It is trivial to check that the residue of (2.9) at the tachyon pole is just  $(A_3)^2$ , as it should.

## 2.2 Closed string

The  $N$ -tachyon vertex for the closed string reads [10]

$$\widehat{V}_{N;0}^{cl} = C_0^{cl} \langle \Omega | \int [dm]_N^0 \exp \left\{ \frac{1}{2} \sum_{i=1}^N \frac{\alpha'}{2\hat{p}_i^2} \ln |V'_i(0)|^2 \right\}$$

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<sup>2</sup>With a different choice in (2.5) of  $\rho$  (which in the case  $N = 3$  is just a constant), a dependence on the momenta would appear in the amplitude via a factor  $\rho^{\sum_i \alpha' p_i^2 - 3}$ .

$$\times \exp \left\{ \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\alpha'}{2} \hat{p}_i \cdot \hat{p}_j \ln |z_i - z_j| \right\}, \quad (2.10)$$

where the measure is

$$[dm]_N^0 = \frac{1}{dV_{abc}} \prod_{i=1}^N \frac{d^2 z_i}{|V'_i(0)|^2}$$

and the  $SL(2, C)$  invariant volume element is:

$$dV_{abc} = \frac{d^2 z_a d^2 z_b d^2 z_c}{|z_a - z_b|^2 |z_a - z_c|^2 |z_c - z_b|^2}.$$

$C_0^{cl}$  is the overall normalization for tree closed string amplitudes. It has the same expression as in the open string case (2.2),  $g_o$  being now substituted by the closed string coupling constant that we will indicate by  $g_c$ . Analogously to the open string case, we define a tachyon state with momentum  $p$  as follows:

$$|p\rangle = \lim_{z, \bar{z} \rightarrow 0} \mathcal{V}_p(z, \bar{z}) |0, p=0\rangle = \mathcal{N}_t e^{ip \cdot \hat{x}} |0, p=0\rangle$$

with  $\mathcal{N}_t$  being a normalization factor given by [11]:

$$\mathcal{N}_t = 2\sqrt{2\pi} g_c (2\alpha')^{(d-2)/4}.$$

Saturating (2.10) with  $N$  tachyon states gives the following result:

$$\mathcal{A}_N^{cl}(p_1, \dots, p_N) = C_0^{cl} \mathcal{N}_t^N \int \frac{1}{dV_{abc}} \prod_{i=1}^N \left[ d^2 z_i |V'_i(0)|^{\frac{\alpha'}{2} p_i^2 - 2} \right] \prod_{\substack{i,j=1 \\ i < j}}^N |z_i - z_j|^{\alpha' p_i \cdot p_j}. \quad (2.11)$$

Once again for on-shell tachyons  $\left(p_i^2 = \frac{4}{\alpha'}\right)$  the local maps cancel out. For arbitrary momenta the same choice of the  $V'_i(0)$ 's made in the open string case guarantees that the integrand in (2.11) is still  $SL(2, C)$  invariant.



For  $N = 3$  the expression in (2.11) specifies in

$$\mathcal{A}_3^{cl} = 8(2\pi)^{3/2} g_c (2\alpha')^{(d-6)/4}. \quad (2.12)$$

Again this is independent of the tachyon momenta.

With the usual choice of the punctures the four tachyon amplitude reads

$$\mathcal{A}_4^{cl} = C_o^{cl} \mathcal{N}_t^4 \int d^2 z |z|^{-\frac{\alpha'}{2}s-4} |1-z|^{-\frac{\alpha'}{2}t-4}. \quad (2.13)$$

The leading Regge trajectory for closed strings is

$$\alpha(s) = \frac{\alpha'}{2}s + 2$$

so, in terms of the Euler beta function, we can write

$$\mathcal{A}_4^{cl} = (2\pi)^2 16 g_c^2 (2\alpha')^{(d-4)/2} B\left(-\frac{1}{2}\alpha(s), -\frac{1}{2}\alpha(t), \frac{1}{2}\alpha(s) + \frac{1}{2}\alpha(t) + 1\right). \quad (2.14)$$

If we restrict to the kinematic shell

$$s + t + u \equiv -\sum_{i=1}^4 p_i^2 = -\frac{16}{\alpha'}, \quad (2.15)$$

we find that

$$\frac{1}{2}\alpha(s) + \frac{1}{2}\alpha(t) = -\frac{1}{2}\alpha(u) - 1$$

so

$$\mathcal{A}_4^{cl} = (2\pi)^2 16 g_c^2 (2\alpha')^{(d-4)/2} B\left(-\frac{1}{2}\alpha(s), -\frac{1}{2}\alpha(t), -\frac{1}{2}\alpha(u)\right). \quad (2.16)$$

This, again, is the same formula one obtains in the on-shell case, with the difference that now the tachyon momenta are only constrained to verify momentum conservation and (2.15).

### 3 Four-tachyon fermionic string amplitudes

#### 3.1 Open fermionic string

In the case of the fermionic string the vertex operator corresponding to a physical state is not unique. In fact one can associate to each physical state an infinite set of vertex operators corresponding to different values of the ghost number, or, equivalently, corresponding to different picture numbers. Nevertheless physical quantities like on-shell scattering amplitudes must be independent of the picture assignment. In order to ensure ghost number conservation on the sphere the picture numbers of the scattering strings must add up to -2. In this paper we will only consider picture numbers -1 and 0. The vertex operator for tachyons in picture 0 is

$$\mathcal{V}_p^{(0)}(z) = p \cdot \psi(z) e^{ip \cdot X(z)},$$

while in picture -1 one has

$$\mathcal{V}_p^{(-1)}(z) = e^{-\phi(z)} e^{ip \cdot X(z)}.$$

The scalar field  $\phi(z)$  has a simple expansion in oscillators given by

$$\phi(z) = \hat{x} + \hat{N} \lg z + \sum_{n \neq 0} \frac{\hat{\alpha}_n}{n} z^{-n},$$

where

$$[\hat{x}, \hat{N}] = 1; \quad [\hat{\alpha}_m, \hat{\alpha}_n] = -m \delta_{m+n, 0}.$$

The zero mode acts on a state in the picture  $a$  according to

$$\hat{N} |\chi\rangle_a = -a |\chi\rangle_a.$$

The  $N$ -tachyon Vertex to be used in this case can be obtained from the  $N$ -string Vertex for the Neveu-Schwarz string in Ref. [9] with a new factor taking into account

the contribution of the scalar  $\phi$  field [10]. The resulting vertex is

$$\begin{aligned} \widehat{V}_{N;0}^{op} = & \langle \Omega | \int [dm]_N^0 \exp \left\{ -\alpha' \sum_{\substack{i,j=1 \\ i \neq j}}^N \ln \frac{\sqrt{V'_i(0) V'_j(0)}}{z_i - z_j} \hat{p}_i \cdot \hat{p}_j \right\} \\ & \times \exp \left\{ -\frac{i}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\sqrt{V'_i(0) V'_j(0)}}{z_i - z_j} b_{1/2}^{(i)} \cdot b_{1/2}^{(j)} - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \ln \frac{\sqrt{V'_i(0) V'_j(0)}}{z_i - z_j} \hat{N}_i \hat{N}_j \right\}, \end{aligned} \quad (3.1)$$

where the measure is defined in (2.3). Notice that here, as in the definition of the tachyon state, we skip the normalization factors which are different from the bosonic case.

It is straightforward to see that saturating  $\widehat{V}_{N;0}^{op}$  with  $N = 3$  tachyons yields zero for any value of the tachyon momenta.

On the other hand, the contribution to the amplitude obtained by saturating this vertex on a four-tachyon state  $|\Omega'\rangle$ , with the tachyons put in some picture  $P_i$ , depends on the particular picture chosen. For example if we choose to put the four tachyons in the picture  $P_1=[0,0,-1,-1]$  we have

$$|\Omega'\rangle = p_1 \cdot b_{-1/2}^{(1)} |0; p_1\rangle_0 \otimes p_2 \cdot b_{-1/2}^{(2)} |0; p_2\rangle_0 \otimes |0; p_3\rangle_{-1} \otimes |0; p_4\rangle_{-1}$$

and the corresponding amplitude is

$$\begin{aligned} \mathcal{A}_4^{op(1)}(p_1, \dots, p_4) = & p_1 \cdot p_2 \int \frac{1}{dV_{abc}} \prod_{i=1}^3 [\vartheta(z_i - z_{i+1})] \prod_{i=1}^4 \left[ dz_i V'_i(0)^{\alpha' p_i^2 - \frac{1}{2}} \right] \\ & \times \prod_{i < j=1}^4 (z_i - z_j)^{2\alpha' p_i \cdot p_j} \frac{1}{(z_1 - z_2)(z_3 - z_4)}. \end{aligned} \quad (3.2)$$

In the fermionic case the on-shell condition for open string tachyons reads

$$p_i^2 = \frac{1}{2\alpha'}$$

so the local map cancellation in the on-shell amplitude is again trivially verified.

It is easy to show that projective invariance for the off-shell amplitude still holds here if  $V'_i(0)$  is given by (2.5). Indeed the expression (3.2) only differs from the bosonic one

by a factor

$$\frac{\prod_{i=1}^4 V'_i(0)^{\frac{1}{2}}}{(z_1 - z_2)(z_3 - z_4)},$$

which can be shown to be projective invariant making use of the relations (2.6). With the usual choice for the fixed punctures we can write the amplitude (3.2) as

$$\begin{aligned} \mathcal{A}_4^{op(1)}(p_1, \dots, p_4) &= p_1 \cdot p_2 \int_0^1 dz z^{-\alpha' s - 2} (1 - z)^{-\alpha' t - 1} \\ &= p_1 \cdot p_2 \frac{\Gamma(-\alpha' s - 1) \Gamma(-\alpha' t)}{\Gamma(-\alpha' s - \alpha' t - 1)}, \end{aligned} \quad (3.3)$$

which manifestly depends on the picture assignment. Furthermore this amplitude exhibits an unwanted singularity for  $s = -1/\alpha'$ . On-shell one has

$$2\alpha' p_1 \cdot p_2 = -\alpha' s - 1$$

and the kinematic factor just cancels the unwanted singularity, but off-shell this is not the case. The solution to this puzzle is that the contribution to the amplitude in each channel is to be obtained by averaging over all possible picture assignments. Then one recovers the expected analyticity properties if condition (1.1) is imposed.

There are six different possible choices of the picture assignments and the corresponding amplitudes are gathered into couples, each of which yielding the same contribution. For example  $\mathcal{A}_4^{op(1)}$  in (3.3) is coupled with the amplitude

$$\mathcal{A}_4^{op(2)} = p_3 \cdot p_4 \frac{\Gamma(-\alpha' s - 1) \Gamma(-\alpha' t)}{\Gamma(\alpha' u + 1)}$$

corresponding to  $P_2 = [-1, -1, 0, 0]$ , to give

$$\mathcal{A}_4^{op(1)} + \mathcal{A}_4^{op(2)} = (p_1 \cdot p_2 + p_3 \cdot p_4) \frac{\Gamma(-\alpha' s - 1) \Gamma(-\alpha' t)}{\Gamma(\alpha' u + 1)}.$$

The kinematic condition (1.1) now reads

$$s + t + u = -\frac{8}{\alpha'} \quad (3.4)$$

ensuring that

$$p_1 \cdot p_2 + p_3 \cdot p_4 = -s - \frac{1}{\alpha'},$$

by which we obtain

$$\mathcal{A}_4^{op(1)} + \mathcal{A}_4^{op(2)} \sim \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(\alpha'u+1)}.$$

Here we have used the standard formula

$$a\Gamma(a) = \Gamma(a+1).$$

The remaining four possible pictures give rise to identical contributions and the total off-shell scattering amplitude can therefore be written as

$$\mathcal{A}_4^{op}(s, t, u) \sim \left[ \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(\alpha'u+1)} + \frac{\Gamma(-\alpha's)\Gamma(-\alpha'u)}{\Gamma(\alpha't+1)} + \frac{\Gamma(-\alpha't)\Gamma(-\alpha'u)}{\Gamma(\alpha's+1)} \right]. \quad (3.5)$$

Also in the fermionic case, therefore, the off-shell four-tachyon amplitude assumes the same form as the corresponding on-shell one. The formula (3.5) is valid for arbitrary  $s, t, u$  provided they satisfy (3.4).

### 3.2 Closed fermionic string

The  $N$ -tachyon vertex can be obtained duplicating the open string vertex:

$$\begin{aligned} \widehat{V}_{N;0}^{cl} = & \langle \Omega | \int [dm]_N^0 \exp \left\{ \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\alpha'}{2} \ln \frac{|z_i - z_j|^2}{|V'_i(0) V'_j(0)|} \hat{p}_i \cdot \hat{p}_j \right\} \\ & \times \exp \left\{ -\frac{i}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\sqrt{V'_i(0) V'_j(0)}}{z_i - z_j} b_{1/2}^{(i)} \cdot b_{1/2}^{(j)} - \frac{i}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\sqrt{\bar{V}'_i(0) \bar{V}'_j(0)}}{\bar{z}_i - \bar{z}_j} \tilde{b}_{1/2}^{(i)} \cdot \tilde{b}_{1/2}^{(j)} \right\} \\ & \times \exp \left\{ \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \ln \frac{\sqrt{V'_i(0) V'_j(0)}}{z_i - z_j} N_i \cdot N_j + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \ln \frac{\sqrt{\bar{V}'_i(0) \bar{V}'_j(0)}}{\bar{z}_i - \bar{z}_j} \tilde{N}_i \cdot \tilde{N}_j \right\}. \end{aligned} \quad (3.6)$$

In this case there are of course 36 possible choices for the picture numbers, which must add to -2 in each sector. Let us consider for example the choice made in [13], that is

$P_1=[(0,0),(0,0),(-1,-1),(-1,-1)]$ . The corresponding amplitude is obtained by saturating (3.6) on the state

$$|\Omega'\rangle = p_1 \cdot b_{-1/2}^{(1)} p_1 \cdot \tilde{b}_{-1/2}^{(1)} |0; p_1\rangle_{0,0} \otimes p_2 \cdot b_{-1/2}^{(2)} p_2 \cdot \tilde{b}_{-1/2}^{(2)} |0; p_2\rangle_{0,0} \otimes |0; p_3\rangle_{-1,-1} \otimes |0; p_4\rangle_{-1,1}.$$

This results in the following expression

$$\begin{aligned} \mathcal{A}_4^{cl(1)}(p_1, \dots, p_4) &= (p_1 \cdot p_2)^2 \int \frac{1}{dV_{abc}} \prod_{i=1}^4 \left[ d^2 z_i |V'_i(0)|^{\alpha' \frac{p_i^2}{2} - 1} \right] \\ &\times \prod_{\substack{i,j=1 \\ i < j}}^4 |z_i - z_j|^{\alpha' p_i \cdot p_j} \frac{1}{|z_1 - z_2|^2 |z_3 - z_4|^2}. \end{aligned} \quad (3.7)$$

The independence of the local coordinate maps is still trivially verified in the on-shell case, i.e. when

$$p_i^2 = \frac{2}{\alpha'},$$

while projective invariance holds also off-shell, as in the open string case, provided that the  $V'_i(0)$ 's are again given by (2.5).

With the standard choice for the punctures the amplitude assumes the form

$$\begin{aligned} \mathcal{A}_4^{cl(1)} &= (p_1 \cdot p_2)^2 \int d^2 z |z|^{-\frac{\alpha'}{2}s-4} |1-z|^{-\frac{\alpha'}{2}t-2} \\ &= (p_1 \cdot p_2)^2 B\left(-\frac{\alpha'}{4}s-1, -\frac{\alpha'}{4}t, \frac{\alpha'}{4}(s+t)+2\right). \end{aligned}$$

Forcing the tachyon momenta to live on the kinematic shell

$$s+t+u \equiv -\sum_{i=1}^4 p_i^2 = -\frac{8}{\alpha'}, \quad (3.8)$$

the amplitude can be rewritten as

$$\begin{aligned} \mathcal{A}_4^{cl(1)} &= (p_1 \cdot p_2)^2 B\left(-\frac{\alpha'}{4}s-1, -\frac{\alpha'}{4}t, -\frac{\alpha'}{4}u\right) \\ &= \pi (p_1 \cdot p_2)^2 \frac{\Gamma(-\frac{\alpha'}{4}s-1)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(\frac{\alpha'}{4}s+2)\Gamma(\frac{\alpha'}{4}t+1)\Gamma(\frac{\alpha'}{4}u+1)}. \end{aligned}$$

Just as in the open string case, the result strongly depends on the picture assignment and has the wrong analyticity properties. However there exist three more pictures yielding

the same ratio of  $\Gamma$  functions, namely  $P_2=[(-1,-1),(-1,-1),(0,0),(0,0)]$ ,  $P_3=[(0,-1),(0,-1),(-1,0),(-1,0)]$  and  $P_4=[(-1,0),(-1,0),(0,-1),(0,-1)]$ , the corresponding amplitudes being

$$\begin{aligned}\mathcal{A}_4^{cl(2)} &= \pi(p_3 \cdot p_4)^2 \frac{\Gamma(-\frac{\alpha'}{4}s-1)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(\frac{\alpha'}{4}s+2)\Gamma(\frac{\alpha'}{4}t+1)\Gamma(\frac{\alpha'}{4}u+1)}, \\ \mathcal{A}_4^{cl(3)} &= \pi(p_1 \cdot p_2)(p_3 \cdot p_4) \frac{\Gamma(-\frac{\alpha'}{4}s-1)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(\frac{\alpha'}{4}s+2)\Gamma(\frac{\alpha'}{4}t+1)\Gamma(\frac{\alpha'}{4}u+1)}, \\ \mathcal{A}_4^{cl(4)} &= \pi(p_1 \cdot p_2)(p_3 \cdot p_4) \frac{\Gamma(-\frac{\alpha'}{4}s-1)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(\frac{\alpha'}{4}s+2)\Gamma(\frac{\alpha'}{4}t+1)\Gamma(\frac{\alpha'}{4}u+1)}.\end{aligned}$$

Summing the four kinematic factors and enforcing (3.8) leads to

$$(p_1 \cdot p_2 + p_3 \cdot p_4)^2 = (s + \frac{4}{\alpha'})^2$$

so that

$$\begin{aligned}\mathcal{A}_4^{cl} &= \mathcal{A}_4^{cl(1)} + \mathcal{A}_4^{cl(2)} + \mathcal{A}_4^{cl(3)} + \mathcal{A}_4^{cl(4)} \\ &\sim \frac{\Gamma(-\frac{\alpha'}{4}s)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(\frac{\alpha'}{4}s+1)\Gamma(\frac{\alpha'}{4}t+1)\Gamma(\frac{\alpha'}{4}u+1)}.\end{aligned}$$

Similar considerations can be used to deal with contributions arising from the other 32 possible picture assignments. They all can be seen to belong to groups of four contributions adding up to the same expression, so that the total amplitude is found to be

$$\mathcal{A}_4^{cl}(s, t, u) \sim \frac{\Gamma(-\frac{\alpha'}{4}s)\Gamma(-\frac{\alpha'}{4}t)\Gamma(-\frac{\alpha'}{4}u)}{\Gamma(\frac{\alpha'}{4}s+1)\Gamma(\frac{\alpha'}{4}t+1)\Gamma(\frac{\alpha'}{4}u+1)}. \quad (3.9)$$

The amplitude (3.9) coincides with the corresponding on-shell one, but the tachyon momenta are only constrained to verify (3.8). This result can be extended to the four-tachyon of type 0 theory and coincides with the one computed in [13].

In conclusion we have computed projective invariant off-shell tachyon amplitudes showing that, with a special choice of the local coordinate systems around the punctures they can be put in the same form as their on-shell counterparts. This has been done for the bosonic string and also for the fermionic string, where little is known from the side of string field theory. Suitable low-energy limits may allow to shed new light on effective actions for tachyons and hence on their dynamics.

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## References

- [1] A. Sen, JHEP 9808 12 (1998).
- [2] See, for instance:
  - V. A. Kostelecky and S. Samuel, *Phys. Lett.* **B207**, 169 (1988);
  - A. Sen, JHEP **9912**, 027 (1999);
  - A. Sen and B. Zwiebach, JHEP **0003**, 002 (2000);
  - N. Berkovits, JHEP **0004**, 022 (2000);
  - A. A. Gerasimov and S. L. Shatashvili, *On Exact Tachyon Potential in Open String Field Theory*, hep-th/0009103;
  - D. Kutasov, M. Mariño and G. Moore, *Some Exact Results on Tachyon Condensation in String Field Theory*, hep-th/0009148.
- [3] A. Cohen, G. Moore, P. Nelson and J. Polchinski, *Nucl. Phys.* **B267**, 143 (1986);  
**B281**, 127 (1987);  
C. Varughese and W. I. Weisberger, *Phys. Rev.* **D 37**, 981 (1988).
- [4] E. Witten, *Nucl. Phys.* **B268**, 253 (1986);  
D. J. Gross and A. Jevicki, *Nucl. Phys.* **B283**, 1 (1987);  
J. H. Sloan, *Nucl. Phys.* **B302**, 349 (1988);  
S. Samuel, *Nucl. Phys.* **B308**, 285 (1988);  
R. Bluhm and S. Samuel, *Nucl. Phys.* **B323**, 337 (1989).
- [5] See, for example, P. Di Vecchia, *Multiloop amplitudes in string theory*, Erice, *Theor. Phys.* (1992), and references therein.
- [6] L. Cappiello, R. Marotta, R. Pettorino and F. Pezzella, *Mod. Phys. Lett. A* **13**, 2433 (1998); **13**, 2845 (1998);



- L. Cappiello, A. Liccardo, R. Marotta, R. Pettorino and F. Pezzella, *Prescriptions for off-shell bosonic string amplitudes*, Proceedings Corfu, Greece 1998, *Lecture Notes in Physics*, vol. 525 (Springer, 1999).
- [7] A. Liccardo, R. Marotta and F. Pezzella, *Mod. Phys. Lett. A* **14**, 799 (1999).
  - [8] T. Kubota and G. Veneziano, *Phys. Lett. B* **207** 419 (1988).
  - [9] A. Clarizia and F. Pezzella, *Nucl. Phys.* **B298** 636 (1987).
  - [10] P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and S. Sciuto, *Nucl. Phys.* **B322**, 123 (1989).
  - [11] P. Di Vecchia, L. Magnea, A. Lerda, R. Russo and R. Marotta, *Nucl. Phys.* **B469**, 235 (1996).
  - [12] C. Lovelace, *Phys. Lett. B* **32**, 703 (1970);  
P. Di Vecchia, R. Nakayama, J. L. Petersen and S. Sciuto, *Nucl. Phys.* **B282**, 179 (1987).
  - [13] I. R. Klebanov and A. A. Tseytlin, *Nucl. Phys.* **B546**, 155 (1999).